Math 2143 - Brief Calculus with Applications

Exam #1 - 2021.02.12

Solutions

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1. (a) State the definition of a function $f(x)$ being continuous at $x = x_0$.

A function $f(x)$ is continuous at $x = x_0$ if and only if $\lim_{x \to x_0} = f(x_0)$.

(b) State the limit formula for the definition of the derivative $f'(x)$ of a function $f(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2. Compute the following limits:

(a) $\lim_{x \to 1} 2x^2 - x - 1$

Since this function $2x^2 - x - 1$ is a polynomial, we can simply plug in the limit

$$\lim_{x \to 1} 2x^2 - x - 1 = 2 \cdot 1^2 - 1 - 1 = 0$$

(b) $\lim_{x \to 1} \frac{2x^2 - x - 1}{x - 1}$

$$\begin{aligned}
\lim_{x \to 1} \frac{2x^2 - x - 1}{x - 1} &= \lim_{x \to 1} \frac{(2x+1)(x-1)}{x-1} \\
&= \lim_{x \to 1} (2x+1) \cdot \frac{(x-1)}{x-1} \\
&= \lim_{x \to 1} (2x+1) \cdot \lim_{x \to 1} \frac{(x-1)}{x-1} \\
&= 3 \cdot 1 \\
&= 3
\end{aligned}$$

(c) $\lim_{h \to 0} \frac{\frac{1}{2h+3} - \frac{1}{3}}{h}$

$$\begin{aligned}
\lim_{h \to 0} \frac{\frac{1}{2h+3} - \frac{1}{3}}{h} &= \lim_{h \to 0} \frac{\frac{3 - (2h+3)}{3(2h+3)}}{h} \\
&= \lim_{h \to 0} \frac{-2h}{3h(2h+3)} \\
&= \lim_{h \to 0} \frac{-2}{3(2h+3)} \\
&= -\frac{2}{9}
\end{aligned}$$

(d) $\lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h}$

$$\begin{aligned}\lim_{h\to 0}\frac{\sqrt{4+h}-2}{h} &= \lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}\cdot\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}\\ &= \lim_{h\to 0}\frac{h}{h\cdot(\sqrt{4+h}+2)}\\ &= \lim_{h\to 0}\frac{h}{h}\cdot\frac{1}{\sqrt{4+h}+2}\\ &= \lim_{h\to 0}\frac{1}{\sqrt{4+h}+2}\\ &= \frac{1}{4}\end{aligned}$$

3. Find the equation of the tangent line to the function $f(x) = 4x^2 - 2x + \frac{1}{x}$ at $x = 1$.

We use the formula $y - y_0 = m(x - x_0)$, where slope is $f'(x_0)$. So we fill in the blanks, $x_0 = 1$, $y_0 = 3$. Next we need to compute the derivative, which is

$$f'(x) = 8x - 2 - \frac{1}{x^2}$$

Plugging in x_0 gives $f'(1) = 5$. So the equation of our tangent line is $y - 3 = 5(x - 1)$.

4. Compute the following derivatives:

(a) $\frac{\mathrm{d}^2}{\mathrm{d}w^2}\left(w^3 - \frac{4}{w^2} + 2\right)$

We use the power rule with the sum/difference rule to compute the first derivative

$$\frac{\mathrm{d}}{\mathrm{d}w}\left(w^3 - \frac{4}{w^2} + 2\right) = 3w^2 + \frac{8}{w^3}$$

Next, we compute the derivative of the above result to get the second derivative using the same rules:

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}w^2}\left(w^3 - \frac{4}{w^2} + 2\right) &= \frac{\mathrm{d}}{\mathrm{d}w}\left(3w^2 + \frac{8}{w^3}\right)\\ &= 6w - \frac{24}{w^4}\end{aligned}$$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\left(7^4\right)$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(7^4\right) = 0$$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{2x^3 - 3\sqrt{x} + 1}$

This is a general power rule derivative:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{2x^3 - 3\sqrt{x} + 1} &= \frac{1}{2\sqrt{2x^3 - 3\sqrt{x} + 1}}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(2x^3 - 3\sqrt{x} + 1\right)\\ &= \frac{1}{2\sqrt{2x^3 - 3\sqrt{x} + 1}}\cdot\left(6x^2 - \frac{3}{2\sqrt{x}}\right)\end{aligned}$$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^3 - 4x^2}{3x^2 - 1}\right)$

Lastly, we use the quotient rule on this problem:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^3-4x^2}{3x^2-1}\right) = \frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3-4x^2\right)\cdot\left(3x^2-1\right)-\left(x^3-4x^2\right)\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(3x^2-1\right)}{\left(3x^2-1\right)^2}$$
$$= \frac{\left(3x^2-8x\right)\cdot\left(3x^2-1\right)-\left(x^3-4x^2\right)\cdot(6x)}{\left(3x^2-1\right)^2}$$

(e) $\frac{\mathrm{d}}{\mathrm{d}x}\left((3x^2-1)^2\cdot\sqrt{2x+1}\right)$

This is the product rule, with the chain rule applied to each derivative in the product.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((3x^2-1)^2\cdot\sqrt{2x+1}\right) = \left(\frac{\mathrm{d}}{\mathrm{d}x}(3x^2-1)^2\right)\cdot\sqrt{2x+1}+(3x^2-1)^2\cdot\left(\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{2x+1}\right)$$
$$= 2\cdot(3x^2-1)\cdot\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(3x^2-1\right)\right)\cdot\sqrt{2x+1}+(3x^2-1)^2\cdot\frac{1}{2}\frac{1}{\sqrt{2x+1}}\cdot\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(2x+1\right)\right)$$
$$= 2\cdot(3x^2-1)\cdot 6x\cdot\sqrt{2x+1}+(3x^2-1)^2\cdot\frac{1}{2}\frac{1}{\sqrt{2x+1}}\cdot 2$$