Math 2143 - Brief Calculus with Applications

Exam #2 - 2021.03.24

Solutions

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1. (a) State the definition of a function $f(x)$ being continuous at $x = x_0$.

A function $f(x)$ is continuous at $x = x_0$ if and only if $\lim_{x \to x_0} f(x) = f(x_0)$.

(b) State the limit formula for the definition of the derivative $f'(x)$ of a function $f(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(c) State the product rule formula for differentiation.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x) \cdot g(x)\right) = f'(x)g(x) + f(x)g'(x)$$

(d) State the quotient rule formula for differentiation.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

(e) State the chain rule formula for differentiation.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)$$

(f) State the domain and range of the function $f(x) = e^x$.

The domain of e^x is $(-\infty, \infty)$ (or $\mathbb{R}$) and the range is $(0, \infty)$.

(g) State the domain and range of the function $f(x) = \ln(x)$.

The domain of $\ln(x)$ is $(0, \infty)$, and the range is $(-\infty, \infty)$.

2. Compute the equation of the tangent line to the function $f(x) = 2x^3 - 3x^2 + 6x - 2$ at $x = -1$.

Remember that the point-slope formula for the equation of a line is $y - y_0 = m(x - x_0)$, where $y_0 = f(-1)$, and $m = f'(-1)$ for our problem. Thus, the equation is: $y - f(-1) = f'(-1)(x - (-1))$. So first, we need to compute $f(-1) = -2 - 3 - 6 - 2 - 13$. Next, $f'(x) = 6x^2 - 6x + 6$, and thus $f'(-1) = 18$. The equation of the tangent line is $y + 13 = 18(x + 1)$.

3. Compute the following derivatives.

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 e^{3x-1} + 7x\right)$

We use the product rule here:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 e^{3x-1} + 7x\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\right) \cdot e^{3x-1} + x^2 \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{3x-1} +\right) + 7 \\ &= 2x \cdot e^{3x-1} + x^2 \cdot e^{3x-1} \cdot \frac{\mathrm{d}}{\mathrm{d}x}(3x-1) + 7 \\ &= 2x \cdot e^{3x-1} + x^2 \cdot e^{3x-1} \cdot 3 + 7\end{aligned}$$

(b) $\frac{\mathrm{d}}{\mathrm{d}z} \frac{\ln(2z)}{z^2+1}$

We use the quotient rule here:

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{\ln(2z)}{z^2+1} = \frac{\frac{\mathrm{d}}{\mathrm{d}z}(\ln(2z))\cdot(z^2+1) - \ln(2z)\cdot\frac{\mathrm{d}}{\mathrm{d}z}(z^2+1)}{(z^2+1)^2}$$
$$= \frac{\frac{\frac{\mathrm{d}}{\mathrm{d}z}(2z)}{2z}\cdot(z^2+1) - \ln(2z)\cdot 2z}{(z^2+1)^2}$$
$$= \frac{\frac{2}{2z}\cdot(z^2+1) - \ln(2z)\cdot 2z}{(z^2+1)^2}$$

(c) $\frac{\mathrm{d}}{\mathrm{d}w}2^{3w-1}$

This is the chain rule, where we have to remember that $\frac{\mathrm{d}}{\mathrm{d}w}2^w = 2^w\ln(2)$:

$$\frac{\mathrm{d}}{\mathrm{d}w}2^{3w-1} = 2^{3w-1}\ln(2)\frac{\mathrm{d}}{\mathrm{d}w}(3w-1)$$
$$= 2^{3w-1}\ln(2)\cdot 3$$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}\left(x\ln(x) - x\right)$

This is another product rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\ln(x) - x\right) = \left(\frac{\mathrm{d}}{\mathrm{d}x}x\right)\cdot\ln(x) + x\cdot\frac{\mathrm{d}}{\mathrm{d}x}\ln(x) - 1$$
$$= 1\cdot\ln(x) + x\cdot\frac{1}{x} - 1$$
$$= \ln(x)$$

5. Solve the following equations for the given variable:

(a) $e^{2x+1} = 4$

Here, we use the fact the natural logarithm is the inverse to the exponential function, and apply ln to both sides:

$$\ln\left(e^{2x+1}\right) = \ln(4)$$

And thus $2x+1 = \ln(4)$. We can now isolate x to get $x = \frac{\ln(4)-1}{2}$.

(b) $\ln(x)+\ln(x-1) = \ln(2)$

We start by using the logarithm rule that $\ln(a)+\ln(b) = \ln(ab)$ to get

$$\ln(x(x-1)) = \ln(2)$$

We can next raise both sides as exponents of e to get

$$e^{\ln(x(x-1))} = e^{\ln(2)}$$

And thus

$$x(x-1) = 2$$

This is a quadratic equation: $x^2 - x - 2 = 0$. This factors as $(x-2)(x+1) = 0$, so the solutions are $x = -1$ and $x = 2$. Note that we cannot have $x = -1$ as a solution as it cannot by plugged into the original equations, as the domain of $\ln(x)$ is $(0,\infty)$.