Math 2143 - Brief Calculus with Applications

Exam #3 - 2021.04.23

Solutions

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1. Consider the function $G(x) = x^3 + 3x^2 - 12$.

(a) Compute $G'(x)$.

$$G'(x) = 3x^2 + 6x$$

(b) Find the critical points of $G(x)$.

Setting $G'(x) = 0$ gives $3x(x+2) = 0$, or $x = 0$ and $x = -2$.

(c) State the intervals of increase and decrease for $G(x)$.

Using the critical points found from part (b), we simply test a point on each interval created by cutting the real line at the critical points. Since $G'(x)$ is a parabola opening upwards, we have that $G'(x) > 0$ ($G(x)$ increasing) for $x \in (-\infty, -2) \cup (0, \infty)$, and $G'(x) < 0$ ($G(x)$ decreasing) for $x \in (-2, 0)$.

(d) Classify the critical points from part (b) using the First Derivative Test.

Since $G(x)$ goes from increasing to decreasing at $x = -2$ this is a local max, similarly at $x = 0$, $G(x)$ goes from decreasing to increasing so $x = 0$ is a local min.

(e) State the intervals of concavity $G(x)$.

Here $G''(x) = 6x + 6$, and setting it to zero gives $x = -1$. This is a line with positive slope, so $G''(x) > 0$ (concave up) for $x > -1$, and concave down for $x \in (-\infty, -1)$.

(f) Locate any inflection points of $G(x)$.

There is a change in concavity at $x = -1$, which is therefore an inflection point.

(g) Classify the critical points from part (b) using the Second Derivative Test.

At $x = -2$ the function is concave down, thus a local maximum. At $x = 0$ the function is concave up, thus a local minimum.

(h) Find the absolute maximum and minimum values of $G(x)$ on the interval $[-3, 1]$.
The absolute maximum and minimum of $G(x)$ on $[-3, 1]$ must occur at an endpoint or at a critical point.

$$G(-3) = -12,\ G(-2)) = -8,\ G(0) = -12.\ G(1) = -8$$

Therefore, the absolute maximum occurs at either $x = -2$ and $x = 1$, with a value of -8, and the absolute minimum occurs at either $x = -3$ and $x = 0$ with a value of -12.

2. Consider the function $F(x) = \dfrac{(2x+1)^2}{(x-1)(x+2)}$.

(a) State the domain of $F(x)$.

The domain is all real numbers except where the denominator is zero, which is $x = 1, -2$, thus the domain is $\mathbb{R} - \{-2, 1\}$.

(b) Locate the roots of $F(x)$.

The roots are where the numerator is zero, which is at $x = -1/2$.

(c) Locate the vertical asymptotes.

Vertical asymptotes are places from part (a) which also are not zeros of the numerator, thus $x = -2$ and $x = 1$ are vertical asymptotes.

(d) Locate (if any) horizontal or slant asymptotes.

Since the degree of the numerator is the same as the degree of the denominator, there is a horizontal asymptote which is the ratio of the coefficients of the leading terms. Thus, we have $y = 4$ is the horizontal asymptote.

(e) Find the y-intercept.

The y-intercept is when $x = 0$, thus $F(0) = -2$ is the intercept.

(f) Compute the limits at the vertical asymptotes.

Since we have horizontal asymptote of $y = 4$, we know:

$$\lim_{x\to 1^+} F(x) = \lim_{x\to -2^-} F(x) = +\infty$$

Since $F(0) = -2$ and the root at $x = -1/2$ has multiplicity 2, we also get:

$$\lim_{x\to 1^-} F(x) = \lim_{x\to -2^+} F(x) = -\infty$$

(g) Sketch the graph of $F(x)$ using the information from parts (a)–(f).

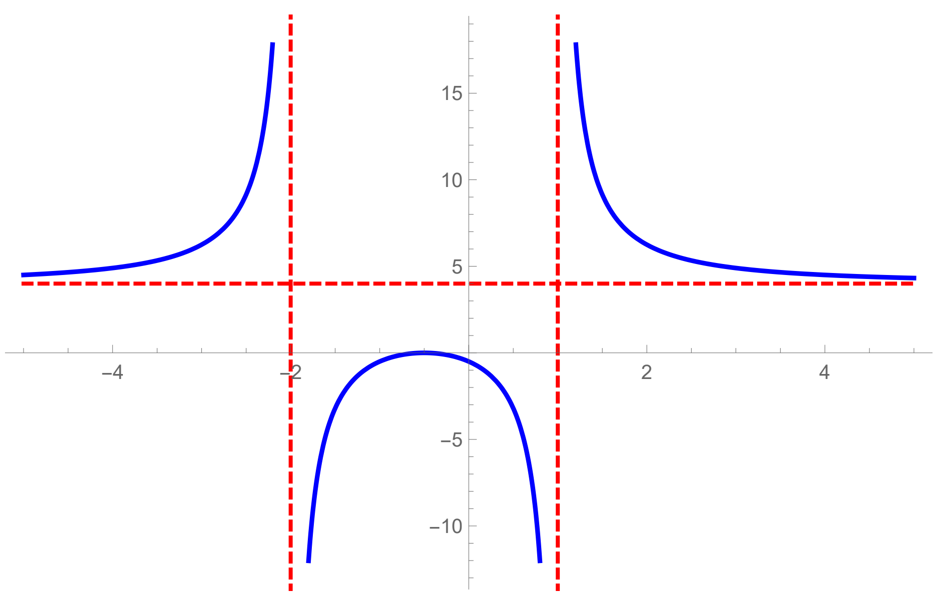

FIGURE 1. Graph of $F(x)$

3. Upon inspection, determine the slant asymptote of the function $R(x) = \dfrac{6x^5 - 3x^4 + 2x^2 + 1}{3x^4 + 4x^3 + 8x^2 + 2x}$

The leading term in the denominator is $3x^4$, so we take all terms of degree 4 or greater in the numerator of $R(x)$: $\frac{6x^5 - 3x^4}{3x^4}$. Simplifying this gives the slant asymptote $y = 2x - 1$.

4. Find $\frac{dy}{dx}$ by implicit differentiation if $3x^2 + 4xy^2 = \ln(2x+1) + 6$.

Applying $\frac{d}{dx}$ treating y as a function of x gives:

$$6x + 4y^2 + 8xy\frac{dy}{dx} = \frac{2}{2x+1}$$

Solving for $\frac{dy}{dx}$ gives

$$\frac{dy}{dx} = \frac{\frac{2}{2x+1} - 6x - 4y^2}{8xy}.$$

5. Use logarithmic differentiation to compute the derivative of the following function:

$$T(x) = \frac{(3x+1)(5x-7)^3(2-6x)^{3/2}}{\sqrt{8x+1}(4x^2+7)}$$

From logarithmic differentiation we have

$$T'(x) = T(x)\frac{d}{dx}\ln(T(x)),$$

where

$$\ln(T(x)) = \ln(3x+1) + 3\ln(5x-7) + \frac{3}{2}\ln((2-6x)) - \frac{1}{2}\ln(8x+1) - \ln(4x^2+7)$$

So

$$T'(x) = T(x)\left[\frac{3}{3x+1} + 3\cdot\frac{5}{5x-7} + \frac{3}{2}\cdot\frac{-6}{2-6x} - \frac{1}{2}\cdot\frac{8}{8x+1} - \frac{8x}{4x^2+7}\right].$$

6. In an attempt to do well on this exam, you attempt to offer up the instructor as a sacrifice to the Gods. To do so, the instructor is bled out and his blood pools on the floor. The blood leaves the instructor's body at constant rate of 3 cm^3/sec and pools in a perfect disk shape 1 cm thick (naturally of course). The volume of a disk of radius r and height h is $V = \pi r^2 h$. At what rate is the radius of the pool of blood growing when the disk is 4 cm in radius?

Since the height is fixed at 1 cm, we have $V(t) = \pi r^2(t)h = \pi r^2(t)$. Taking a derivative gives

$$\frac{dV}{dt} = 2\pi r(t)r'(t)$$

Since the volume of blood is flowing into the pool at a constant rate and we want to know the rate at which the radius is growing when $r = 4$ cm, we have

$$3 = 2\pi \cdot 4r'(t)$$

Solving for $r'(t)$ gives $r'(t) = \frac{3}{8\pi}$ cm/sec.