Math 2315 - Calculus 2

Cumulative Quiz #1 - 2021.01.22

Solutions

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1. Compute the following limit: $\lim_{x \to 0} \frac{\sin(3x^2)}{5x^2}$

$$\begin{aligned} \lim_{x \to 0} \frac{\sin(3x^2)}{5x^2} &= \frac{3}{5} \cdot \lim_{x \to 0} \frac{\sin(3x^2)}{3x^2} \\ &= \frac{3}{5} \cdot 1 \\ &= \frac{3}{5} \end{aligned}$$

2. Compute the following derivative: $\frac{\mathrm{d}}{\mathrm{d}x} \cos^3(x^2)$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} \cos^3(x^2) &= 3\cos^2(x^2) \frac{\mathrm{d}}{\mathrm{d}x} \cos(x^2) \\ &= 3\cos^2(x^2) \cdot \left(-\sin(x^2) \frac{\mathrm{d}}{\mathrm{d}x} x^2 \right) \\ &= 3\cos^2(x^2) \cdot \big(-\sin(x^2) \cdot 2x \big) \\ &= -6x\cos^2(x^2) \cdot \sin(x^2) \end{aligned}$$

3. Compute $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the implicitly defined function: $y^2 x + y^2 = x^2 - x$

First we apply $\mathrm{d}/\mathrm{d}x$ to both sides of the equation:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} \left(y^2 x + y^2 \right) &= \frac{\mathrm{d}}{\mathrm{d}x} \left(x^2 - x \right) \\ 2yy'x + y^2 + 2yy' &= 2x - 1 \\ y'(2xy + 2y) &= 2x - 1 - y^2 \\ y' &= \frac{2x - 1 - y^2}{2xy + 2y} \end{aligned}$$

4. Compute the following indefinite integral: $\int \cos^3(4z) \sin(4z)\, \mathrm{d}z$

We perform a substitution here, let $u = \cos(4z)$, then $\mathrm{d}u = -4\sin(4z)\mathrm{d}z$.

$$\begin{aligned} \int \cos^3(4z) \sin(4z)\, \mathrm{d}z &= -\frac{1}{4} \int u^3\, \mathrm{d}u \\ &= -\frac{1}{16} u^4 + C \\ &= -\frac{1}{16} \cos^4(4z) + C \end{aligned}$$

5. Compute the following derivative: $\frac{\mathrm{d}}{\mathrm{d}w} \int_{2w}^{3w^2 - 2w + 1} \tan(t + 1)\, \mathrm{d}t$

We use the Fundamental Theorem of Calculus first, and let $F(t)$ be the function such that $F'(t) = \tan(t + 1)$. Then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}w} \int_{2w}^{3w^2 - 2w + 1} \tan(t + 1)\, \mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}w} \big(F(3w^2 - 2w + 1) - F(2w) \big) \\ &= F'(3w^2 - 2w + 1) \cdot (6w - 2) - F'(2w) \cdot 2 \\ &= \tan(3w^2 - 2w + 2) \cdot (6w - 2) - \tan(2w + 1) \cdot 2 \end{aligned}$$

6. Use the Fundamental Theorem of Calculus to compute the area between the curve and x-axis for $f(x) = x^3 - x^2 - 2x$ on the interval $[-1, 2]$.

$$\begin{aligned}\mathcal{A} &= \int_{-1}^{2} x^3 - x^2 - 2x \, \mathrm{d}x \\ &= \frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2 \Big|_{-1}^{2} \\ &= \left(4 - \frac{8}{3} - 4\right) - \left(\frac{1}{4} + \frac{1}{3} - 1\right) \\ &= -\frac{9}{4}\end{aligned}$$

7. Evaluate $\int x(x^2+1)^{1/4} \, \mathrm{d}x$.

We use the substitution $u = x^2 + 1$, and so $\mathrm{d}u = 2x \, \mathrm{d}x$. Solving for $x \, \mathrm{d}x$ gives $x \, \mathrm{d}x = \frac{1}{2} \, \mathrm{d}u$.

$$\begin{aligned}\int x(x^2+1)^{1/4} \, \mathrm{d}x &= \frac{1}{2}\int u^{1/4} \, \mathrm{d}u \\ &= \frac{1}{2} \cdot \frac{4}{5} u^{5/4} + \mathcal{C} \\ &= \frac{2}{5} \left(x^2+1\right)^{5/4} + \mathcal{C}\end{aligned}$$

8. Evaluate $\int x(x+1)^{1/4} \, \mathrm{d}x$.

We use the substitution $u = x + 1$, and so $\mathrm{d}u = \mathrm{d}x$. Solving for x gives $x = u - 1$.

$$\begin{aligned}\int x(x+1)^{1/4} \, \mathrm{d}x &= \int (u-1)u^{1/4} \, \mathrm{d}u \\ &= \int u^{5/4} - u^{1/4} \, \mathrm{d}u \\ &= \frac{4}{9}u^{9/4} - \frac{4}{5}u^{5/4} + \mathcal{C} \\ &= \frac{4}{9}(x+1)^{9/4} - \frac{4}{5}(x+1)^{5/4} + \mathcal{C}\end{aligned}$$

9. State the definition of a function $f(x)$ being continuous at $x = a$.

A function is continuous at $x = a$ iff $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a)$.

10. Express the average value of the function $f(x)$ on the interval $[a, b]$ in terms of a definite integral.

$$f_{\text{ave}} = \frac{1}{b-a} \int_{a}^{b} f(x) \, \mathrm{d}x$$