Math 2315 - Calculus 2

Cumulative Quiz #2 - 2021.02.08

Solutions

---

1. Compute the following integral: $\displaystyle\int \frac{\sin(\ln(2x))}{x}\,\mathrm{d}x$

   We perform a u-substitution: $u = \ln(2x)$, and $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$.

$$\begin{aligned}\int \frac{\sin(\ln(2x))}{x}\,\mathrm{d}x &= \int \sin(u)\,\mathrm{d}u \\ &= -\cos(u) + \mathcal{C} \\ &= -\cos(\ln(2x)) + \mathcal{C}\end{aligned}$$

2. Derive the formula for $\frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x)$ by the method of implicit differentiation.

   If we start with $y = f^{-1}(x)$, then $f(y) = x$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} f(y) &= \frac{\mathrm{d}}{\mathrm{d}x} x \\ f'(y)\frac{\mathrm{d}y}{\mathrm{d}x} &= 1 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{f'(y)} \\ &= \frac{1}{f'\left(f^{-1}(x)\right)}\end{aligned}$$

3. Compute the following integral: $\displaystyle\int 3x\tanh(x^2)\,\mathrm{d}x$

   We perform a u-substitution: $u = x^2$, and $\mathrm{d}u = 2x\,\mathrm{d}x$, and thus $3x\,\mathrm{d}x = \frac{3}{2}\,\mathrm{d}u$.

$$\begin{aligned}\int 3x\tanh(x^2)\,\mathrm{d}x &= \frac{3}{2}\int \tanh(u)\,\mathrm{d}u \\ &= \frac{3}{2}\int \frac{\sinh(u)}{\cosh(u)}\,\mathrm{d}u \\ &= \frac{3}{2}\ln\left(|\cosh(u)|\right) + \mathcal{C} \\ &= \frac{3}{2}\ln\left(\left|\cosh(x^2)\right|\right) + \mathcal{C} \\ &= \frac{3}{2}\ln\left(\cosh(x^2)\right) + \mathcal{C}\end{aligned}$$

On the last line above, since $\cosh(z) \geq 1$ for all z, the absolute value can be removed.

4. Compute the following integral: $\displaystyle\int \frac{9x}{x^2\sqrt{1-x^4}}\,\mathrm{d}x$

   We perform a u-substitution: $u = x^2$, and $\mathrm{d}u = 2x\,\mathrm{d}x$, and thus $9x\,\mathrm{d}x = \frac{9}{2}\,\mathrm{d}u$.

$$\begin{aligned}\int \frac{9x}{x^2\sqrt{x^4-1}}\,\mathrm{d}x &= \frac{9}{2}\int \frac{1}{u\sqrt{1-u^2}}\,\mathrm{d}u \\ &= -\frac{9}{2}\operatorname{sech}^{-1}(u) + \mathcal{C} \\ &= -\frac{9}{2}\operatorname{sech}^{-1}(x^2) + \mathcal{C}\end{aligned}$$

5. State the *domains* and *ranges* of the three functions $\sin^{-1}(x)$, $\cos^{-1}(x)$ and $\tan^{-1}(x)$.

Function	Domain	Range
$\sin^{-1}(x)$	$[-1,1]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
$\cos^{-1}(x)$	$[-1,1]$	$[0,\pi]$
$\tan^{-1}(x)$	$(-\infty,\infty)$	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

6. Evaluate each of the following:

(a) $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

(b) $\tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$

(c) $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$

7. Compute the following derivative: $\frac{\mathrm{d}}{\mathrm{d}z}\operatorname{sech}^{-1}(\cos(z))$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\operatorname{sech}^{-1}(\cos(z)) &= -\frac{1}{\cos(z)\sqrt{1-\cos^2(z)}}\frac{\mathrm{d}}{\mathrm{d}z}\cos(z)\\ &= \frac{1}{\cos(z)\sqrt{1-\cos^2(z)}}\sin(z)\end{aligned}$$

8. Determine the of values z such that $\frac{1}{\cos(z)\sqrt{1-\cos^2(z)}}\sin(z) = \sec(z)$.

First we note that $\sin^2(z) = 1-\cos^2(z)$, so

$$\begin{aligned}\frac{1}{\cos(z)\sqrt{1-\cos^2(z)}}\sin(z) &= \frac{\sin(z)}{\cos(z)\,|\sin(z)|}\\ &= \frac{\sin(z)}{|\sin(z)|}\sec(z)\end{aligned}$$

We simply need to know where $\sin(z) > 0$, which works for $z \in (0,\pi)$, $(2\pi, 3\pi)$ etc... as well we need to ensure that $\sec(z) \neq 0$.

9. Compute the following limit:

$$\begin{aligned}\lim_{x\to 1}\frac{x-1}{\sqrt{10-x}-3} &= \lim_{x\to 1}\frac{x-1}{\sqrt{10-x}-3}\cdot\frac{\sqrt{10-x}+3}{\sqrt{10-x}+3}\\ &= \lim_{x\to 1}\frac{(x-1)(\sqrt{10-x}+3)}{1-x}\\ &= \lim_{x\to 1}\frac{-(1-x)}{1-x}(\sqrt{10-x}+3)\\ &= \lim_{x\to 1}\frac{-(1-x)}{1-x}\cdot\lim_{x\to 1}(\sqrt{10-x}+3)\\ &= -1\cdot 6\\ &= -6.\end{aligned}$$

10. Compute the following integral: $\int_0^{\frac{\pi}{4}} \tan^3(x)\sec^2(x)\,\mathrm{d}x$

We let $u = \tan(x)$, so $\mathrm{d}u = \sec^2(x)\ \mathrm{d}x$. When $x = 0,\ u = 0$ and when $x = \frac{\pi}{4},\ u = 1$. So

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \tan^3(x)\sec^2(x)\ \mathrm{d}x &= \int_0^1 u^3\ \mathrm{d}u \\ &= \frac{1}{4}u^4\Big|_0^1 \\ &= \frac{1}{4}\left(1-0\right) \\ &= \frac{1}{4}\end{aligned}$$