Math 2315 - Calculus 2

Cumulative Quiz #4 - 2021.04.01

Solutions

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1. Compute the following limit: $\lim_{n\to\infty} n^{a/n}$, for $a > 0$.

   Here this is of the form ∞^0 which is prime form for L'Hˆôpital. Setting $y = n^{a/n}$, the $\ln(y) = \frac{a}{n}\ln(n)$.

$$\begin{aligned}\lim_{n\to\infty} \frac{a}{n}\ln(n) &= \lim_{n\to\infty} \frac{a\ln(n)}{n} \\ &= \lim_{n\to\infty} \frac{a/n}{1} \\ &= 0\end{aligned}$$

Thus $\ln(y) = 0$, and therefore solving for y gives $\lim_{n\to\infty} n^{a/n} = 1$.

2. Determine whether the following series converge or diverge:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{3k^2-1}{4k^2+2}$

   Since $\displaystyle\lim_{k\to\infty} \frac{3k^2-1}{4k^2+2} = \frac{3}{4}$, the series must diverge.

(b) $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{3k-1}{4k^2+2}$

   This is an alternating series with $\displaystyle\lim_{k\to\infty} \frac{3k-1}{4k^2+2} = 0$, thus the series converges.

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{3k+\cos(k)}{4k^2+2}$

   We can do a limit comparison test here, since

$$\frac{2k}{4k^2+2} \leq \frac{3k+\cos(k)}{4k^2+2},$$

   and we know that $\sum_{k=1}^{\infty} \frac{2k}{4k^2+2}$ diverges, we can conclude that the series in question diverges.

(d) $\displaystyle\sum_{k=1}^{\infty} \frac{e^k}{3^{k-1}}$

   We can rewrite the terms in the following manner:

$$\frac{e^k}{3^{k-1}} = 3 \cdot \frac{e^k}{3^k},$$

and thus our series becomes

$$\sum_{k=1}^{\infty} \frac{e^k}{3^{k-1}} = \sum_{k=1}^{\infty} 3 \cdot \frac{e^k}{3^k} = 3 \cdot \sum_{k=1}^{\infty} \left(\frac{e}{3}\right)^k,$$

which is a geometric series with $|r| = \dfrac{e}{3} < 1$, which means the series converges.

(e) $\displaystyle\sum_{k=1}^{\infty} \frac{10^k}{k!}$

First, we can recognize this as the power series expansion for e^{10}, for which convergence is known, or we can apply the ratio test with $a_k = \dfrac{10^k}{k!}$

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\frac{10^{k+1}}{(k+1)!}\cdot\frac{k!}{10^k}\\ &= \lim_{k\to\infty}\frac{10}{k+1}\\ &= 0\end{aligned}$$

Since the limit is zero, the series converges.

(f) $\displaystyle\sum_{k=1}^{\infty}\left(\frac{9+99k}{10+100k}\right)^k$

This is a series for which the root test will work well:

$$\begin{aligned}\lim_{k\to\infty}\left[\left(\frac{9+99k}{10+100k}\right)^k\right]^{1/k} &= \lim_{k\to\infty}\left(\frac{9+99k}{10+100k}\right)\\ &= \frac{99}{100}\end{aligned}$$

Since this limit is less than 1 the series converges.

3. Compute the following integrals:

(a) $\displaystyle\int\cos(w)\sin^2(w)\,\mathrm{d}w$

Here we let $u = \sin(w)$, and thus $du = \cos(w)\,\mathrm{d}w$. So the integral becomes:

$$\begin{aligned}\int\cos(w)\sin^2(w)\,\mathrm{d}w &= \int u^2\,\mathrm{d}u\\ &= \frac{1}{3}u^3 + \mathcal{C}\\ &= \frac{1}{3}\sin^3(w) + \mathcal{C}\end{aligned}$$

(b) $\displaystyle\int\frac{1}{\sqrt{x^2-9}}\,\mathrm{d}x$

Here we let $x = 3\sec(\theta)$, and thus $dx = 3\sec(\theta)\tan(\theta)\,\mathrm{d}\theta$. So the integral becomes:

$$\begin{aligned}\int\frac{1}{\sqrt{x^2-9}}\,\mathrm{d}x &= \int\frac{3\sec(\theta)\tan(\theta)}{\sqrt{9\sec^2(\theta)-9}}\,\mathrm{d}\theta\\ &= \int\frac{3\sec(\theta)\tan(\theta)}{3\sqrt{\sec^2(\theta)-1}}\,\mathrm{d}\theta\\ &= \int\frac{3\sec(\theta)\tan(\theta)}{3\sqrt{\tan^2(\theta)}}\,\mathrm{d}\theta\\ &= \int\frac{\sec(\theta)\tan(\theta)}{\tan(\theta)}\,\mathrm{d}\theta\\ &= \int\sec(\theta)\,\mathrm{d}\theta\\ &= \ln\left(|\sec(\theta)+\tan(theta)|\right) + \mathcal{C}\\ &= \ln\left(\left|\frac{x}{3}+\frac{\sqrt{x^2-9}}{3}\right|\right) + \mathcal{C}\end{aligned}$$

Here we used the fact that if $x = 3\sec(\theta)$, then $\sec(\theta) = \frac{x}{3}$ and $\tan(\theta) = \frac{\sqrt{x^2-9}}{3}$.