Math 2315 - Calculus 2

Cumulative Quiz #5 - 2021.04.19

Solutions

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1. Compute the first 4 terms in the Taylor's series of $f(x) = \sqrt{1+x}$ at $x = 3$.

   First, $f(0) = 2$. We need to compute the first three derivatives of $f(x)$:

$$f'(x) = \frac{1}{2}\frac{1}{\sqrt{1+x}}$$
$$f''(x) = -\frac{1}{4}\frac{1}{(\sqrt{1+x})^3}$$
$$f'''(x) = \frac{3}{8}\frac{1}{(\sqrt{1+x})^5}$$

Now we evaluate these derivatives at $x = 3$ to get: $f'(3) = \frac{1}{2}$, $f''(3) = -\frac{1}{32}$ and $f'''(3) = \frac{3}{256}$. The Taylor series at $x = 3$ for $f(x)$ is given by

$$T_3(x) = 2 + \frac{1}{2}(x-3) - \frac{1}{2}\cdot\frac{1}{32}(x-3)^2 + \frac{1}{6}\cdot\frac{3}{256}(x-3)^3$$

2. Find the Maclaurin series for $f(x) = e^{-2x}$.

   Remember that

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k} x^k,$$

and thus replacing $x =$ with $-x$ gives

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k}(-2x)^k = \sum_{k=0}^{\infty} \frac{(-2)^k}{k} x^k$$

3. Find all values of x for which the following series converges:

$$\sum_{k=1}^{\infty} \frac{1}{k^2}(3x+2)^k$$

   We can perform the ratio test here:

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\left|\frac{1}{(k+1)^2}(3x+2)^(k+1)\cdot\frac{k^2}{(3x+2)^k}\right| \\
&= \lim_{k\to\infty}\frac{k^2}{(k+1)^2}\,|(3x+2)| \\
&= \lim_{k\to\infty}\frac{k^2}{(k+1)^2}\cdot\lim_{k\to\infty}|(3x+2)| \\
&= 1\cdot|(3x+2)|
\end{aligned}$$

For this series to converge, we need $|(3x+2)| < 1$, or $-1 < 3x+2 < 1$. Solving for x gives $x \in \left(-1, -\frac{1}{3}\right)$. All that is left now is to check the two endpoints. Note that if $x = -1$, we have the series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2},$$

which is alternating and the terms go to zero, thus the series converges. Similarly, if $x = -\frac{1}{3}$, we have the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2},$$

which we know converges (p-series with $p = 2$). Thus, the original series converges for $x \in \left[-1, -\frac{1}{3}\right]$.

4. Consider the parametric equation $(x(t), y(t)) = \left(e^t, 4e^{2t}\right)$ for $t \in (-\infty, 1)$.

(a) Sketch a graph of the curve for $t \in (-\infty, 1)$.

Note that as $t \to -\infty$, $(x(t), y(t)) \to (0, 0)$. Also, we have that $y(t) = 4x^2(t)$, thus this is the graph of a parabola, drawn starting at the origin (at $t = 0$) and going to the right as t increases until $t = 1$, when the value is $(e, 4e^2)$.

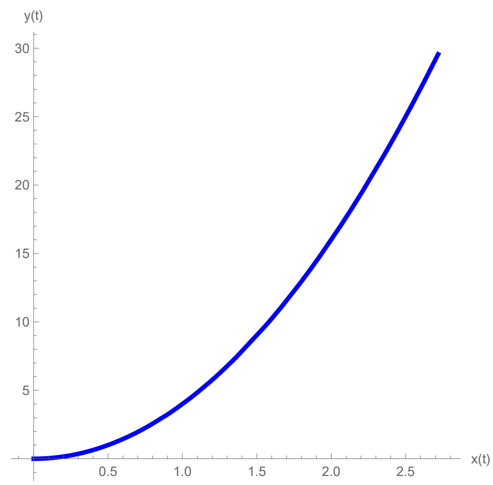

FIGURE 1. Graph of the parametric curve $x(t), y(t)) = \left(e^t, 4e^{2t}\right)$.

(b) Find the slope of the tangent line of the parametric curve when $t = 3$.

First, we compute $\mathrm{d}y/\mathrm{d}t$ and $\mathrm{d}x/\mathrm{d}t$:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 8e^{2t}, \frac{\mathrm{d}x}{\mathrm{d}t} = e^t.$$

Evaluating these at $t = 3$ and taking the ratio gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{8e^{2t}}{e^t} = 8e^t.$$

Evaluating this at $t = 3$ gives a slope of $m = 8e^3$.

5. Consider the parametric equation $(x(t), y(t)) = (\cos(t), 2\sin(2t))$ for $t \in [0, 2\pi]$.

(a) Sketch a graph of the curve for $t \in [0, 2\pi]$.

Note that the curve is periodic with period 2π. The x-coordinates obtain their maximum absolute values when $t = 0, \pi, 2\pi$, and is zero when $t = \pi/2, 3\pi/2$. The y-coordinate is zero at 5 points: $t = 0, \pi/2, \pi, 3\pi/2, 2\pi$, while at a maximum value of 2 when $t = \pi/4, 3\pi4, 5\pi/4, 7\pi/4$. This information should be sufficient to put the graph together.

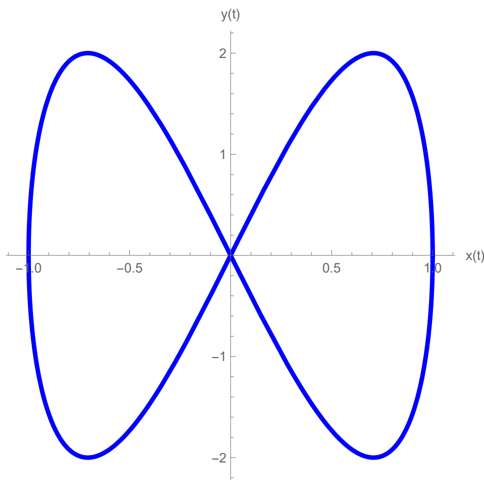

FIGURE 2. Graph of the parametric curve $x(t), y(t)) = (\cos(t), 2\sin(2t))$.

(b) Locate all points (x, y) on the graph where the function will have a vertical OR horizontal tangent line.

For horizontal tangent lines, we set $y'(t) = 0$, where $y'(t) = 4\cos(2t)$. Solving for t gives $t = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$, which are the places where $y(t)$ attains its maximum absolute value. These points are $\left(\pm\frac{1}{\sqrt{2}}, \pm 2\right)$.

For the vertical tangent lines, we set $x'(t) = 0$, where $x'(t) = -\sin(t)$. This occurs when $t = 0, \pi, 2\pi$. These are the points $(1, 0)$ and $(0, 1)$.

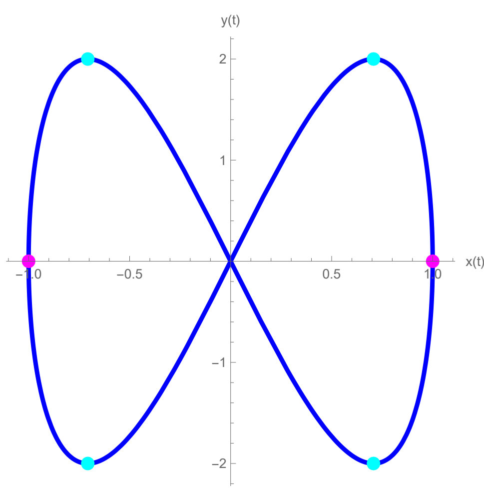

FIGURE 3. Graph of the parametric curve $x(t), y(t)) = (\cos(t), 2\sin(2t))$ along with the points at which there is a vertical tangent line (magenta) and horizontal tangent line (cyan).

6. Compute the following integrals.

(a) $\displaystyle\int \frac{6u^3}{u+1}\,\mathrm{d}u$

We can perform long division or use a substitution, but long division is easier:

$$\begin{aligned}\int \frac{6u^3}{u+1}\,\mathrm{d}u &= 6\int u^2 - u + 1 - \frac{1}{u+1}\,\mathrm{d}u \\ &= 2u^3 - 3u^2 + 6u - 6\ln(|u+1|) + \mathcal{C}\end{aligned}$$

(b) $\displaystyle\int \sin(\ln(t))\,\mathrm{d}t$

This is integration by parts, where $f' = 1$ and $g = \sin(\ln(t))$. Then $f = t$ and $g' = \cos(\ln(t))\frac{1}{t}$. Thus

$$\begin{aligned}\int \sin(\ln(t))\,\mathrm{d}t &= t\,\sin(\ln(t)) - \int t\,\cos(\ln(t))\frac{1}{t}\,\mathrm{d}t \\ &= t\,\sin(\ln(t)) - \int \cos(\ln(t))\,\mathrm{d}t\end{aligned}$$

Applying the same technique to the integral on the right hand side, we set $f' = 1$ and $g = \cos(\ln(t))$, where $f = t$ and $g' = -\sin(\ln(t))\frac{1}{t}$.

$$\begin{aligned}\int \sin(\ln(t))\,\mathrm{d}t &= t\,\sin(\ln(t)) - \int \cos(\ln(t))\,\mathrm{d}t \\ &= t\,\sin(\ln(t)) - \left[t\,\cos(\ln(t)) + \int \sin(\ln(t))\,\mathrm{d}t\right] \\ &= t\,\sin(\ln(t)) - t\,\cos(\ln(t)) - \int \sin(\ln(t))\,\mathrm{d}t\end{aligned}$$

Note that the integral on the left appears on the right as well. Moving the integral on the right to the left side, we have

$$2\int \sin(\ln(t))\,\mathrm{d}t = t\,\sin(\ln(t)) - t\,\cos(\ln(t)).$$

Dividing by 2 gives the integral

$$\int \sin(\ln(t))\,\mathrm{d}t = \frac{t}{2}\,(\sin(\ln(t)) - \cos(\ln(t))) + \mathcal{C}.$$

(c) $\displaystyle\int \sin^3(\theta)\cos^6(\theta)\,\mathrm{d}\theta$

Since the power of $\sin(\theta)$ is odd, we let $u = \cos(\theta)$. Then $\mathrm{d}u = -\sin(\theta)\mathrm{d}\theta$. Also, remember that $\sin^2(\theta) = 1 - \cos^2(\theta) = 1 - u^2$.

$$\begin{aligned}\int \sin^3(\theta)\cos^6(\theta)\,\mathrm{d}\theta &= -\int (1-u^2)u^6\,\mathrm{d}u \\ &= \int u^8 - u^6\,\mathrm{d}u \\ &= \frac{1}{9}u^9 - \frac{1}{7}u^7 + \mathcal{C} \\ &= \frac{1}{9}\cos^9(\theta) - \frac{1}{7}\cos^7(\theta) + \mathcal{C}\end{aligned}$$