Math 3283 - Foundations of Mathematics

Exam #1 - 2021.09.13

Solutions

The Sheffer Stroke (nand) is defined to be the negation of the conjunction, $\sim(p \wedge q)$, and is denoted by the symbol $\uparrow$. The fundamental truth table for $p \uparrow q$ is given by:

p	q	$p \uparrow q$
T	T	F
T	F	T
F	T	T
F	F	T

Note that by definition $p \uparrow q \Leftrightarrow \sim(p \wedge q)$. Use this definition to answer problems 1 through 3.

1. Prove by the method of truth tables that $p \uparrow p \Leftrightarrow \sim p$.

p	$\sim p$	$p \uparrow q$
T	F	F
F	T	T

2. Prove by the method of truth tables that $p \wedge q \Leftrightarrow (p \uparrow q) \uparrow (p \uparrow q)$.

p	q	$p \uparrow q$	$(p \uparrow q) \uparrow (p \uparrow q)$
T	T	F	T
T	F	T	F
F	T	T	F
F	F	T	F

3. Through a string of equivalencies, express $p \to q$ in terms of $\uparrow$ only.

$$\begin{aligned} p \to q &\Leftrightarrow \sim p \vee q \\ &\Leftrightarrow \sim (p \wedge \sim q) \\ &\Leftrightarrow \sim (p \wedge \sim q) \\ &\Leftrightarrow\ p \uparrow \sim q \\ &\Leftrightarrow\ p \uparrow (q \uparrow q)) \end{aligned}$$

4. Determine if each of the following propositions are true or false. Justify your answers!

(a) If $2 \cdot 3 > 10$ or $6 \cdot 3 < 20$, then $6 + 3 = 9$ and $2 + 3 = 5$.

This statement is of the form $(F \vee T) \to (T \wedge T)$, which is $T \to T$ which is true.

(b) $2 \cdot 3 > 10$ iff $2 > 0$ and $3 > 0$.

This statement is of the form $F \longleftrightarrow T$ which is false.

5. Consider the following argument form:

$$\begin{array}{l} p \vee q \\ \sim q \\ \underline{p \to r} \\ \therefore r \end{array}$$

(a) Verify that the argument form is valid by the method of truth tables.

			$\Diamond$	$\heartsuit$	$\triangle$	$\Box$	
p	q	r	$p \vee q$	$\sim q$	$p \to r$	$\Diamond \wedge \heartsuit \wedge \triangle$	$\Box \to r$
T	T	T	T	F	T	F	T
T	T	F	T	F	F	F	T
T	F	T	T	T	T	T	T
T	F	F	T	T	F	F	T
F	T	T	T	F	T	F	T
F	T	F	T	F	T	F	T
F	F	T	F	T	T	F	T
F	F	F	F	T	T	F	T

(b) Using equivalencies and the simple argument forms of MP, MT, and HS, verify that the argument is valid.

First we have that $p \vee q \Leftrightarrow q \vee p$, and also $q \vee p \Leftrightarrow \sim q \to p$. Using Modus Ponens with $\sim q \to p$ and $\sim q$ gives us p. Using p with $p \to r$ along with Modus Ponens gives us r.

6. Determine if each of the following quantified propositions are true of false. For parts (a)–(c), let U be the set of all human beings planet who have lived or are currently living. Justify your answers.

(a) $\forall x \in U\ \exists y \in U\ y$ is the mother of x

This statement is true, every person x has a mother y.

(b) $\exists x \in U\ \forall y \in U\ y$ is the mother of x

This statement is false, there is not someone person x who has everyone as their mother.

(c) $\forall x \in U\ \exists y \in U\ x$ is the mother of y

This statement is false, not everyone person x is the mother of a child.

(d) $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ y = x(x-1)$

This statement is true, given x, we simply let $y = x(x-1)$ as given by the equation.

(e) $\forall y \in \mathbb{R}\ \exists x \in \mathbb{R}\ y = x(x-1)$

This statement is false. Since $y = x(x-1)$ is a parabola opening upwards with roots at $x = 0$ and $x = 1$, it has a vertex at $x = 1/2$ and $y = -1/4$, so if we let $y < -1/4$, no x value will be found to satisfy the equation.

(f) $\forall y \in \mathbb{R}^+\ \exists! x \in \mathbb{R}\ y = x(x-1)$

This statement is false, since we have a parabola opening upwards, we will get two x values for each $y > 0 > -1/4$.