# Math 3283 - Foundations of Mathematics

**Exam #2 - 2021.10.25**

**Solutions**

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1. Prove or disprove: $\forall\, n \in \mathbb{Z}\ 3n^2 - 5n + 6$ is even.

We prove this by cases:

Case (1): $n$ is even. If $n$ is even, then $\exists k \in \mathbb{Z}$ such that $n = 2k$, so

$$\begin{aligned} 3n^2 - 5n + 6 &= 3 \cdot (2k)^2 - 5(2k) + 6 \\ &= 3 \cdot 4k^2 - 10k + 6 \\ &= 12k^2 - 10k + 6 \\ &= 2(6k^2 - 5k + 3) \\ &= 2m \end{aligned}$$

where $m = 6k^2 - 5k + 3 \in \mathbb{Z}$, thus $3n^2 - 5n + 6$ is even.

Case (2): $n$ is odd. If $n$ is odd, then $\exists k \in \mathbb{Z}$ such that $n = 2l + 1$, so

$$\begin{aligned} 3n^2 - 5n + 6 &= 3 \cdot (2l+1)^2 - 5(2l+1) + 6 \\ &= 3 \cdot (4l^2 + 4l + 1) - 10l - 5 + 6 \\ &= 12l^2 + 12l + 3 - 10l - 5 + 6 \\ &= 12l^2 + 2l + 4 \\ &= 2(6l^2 + l + 2) \\ &= 2p \end{aligned}$$

where $p = 6l^2 + l + 2 \in \mathbb{Z}$, thus $3n^2 - 5n + 6$ is even.

2. Prove or disprove: $\forall\, n \geq 3, n \in \mathbb{Z}$ if $n$ is prime, then $n + 1$ is not prime.

Assume that $n$ is prime, and $n \geq 3$. By the definition of $n$ being prime, $2 \nmid n$, thus $n$ is odd, which means $n + 1$ is even, therefore $2 \nmid n + 1$, thus $n + 1$ is not prime.

3. Prove $\forall\, n \geq 2, n \in \mathbb{Z},\ \displaystyle\prod_{k=2}^{n} \frac{k-1}{k+1} = \frac{2}{n(n+1)}$.

We prove this by induction, with base case n=2:

$$\prod_{k=2}^{2} \frac{k-1}{k+1} = \frac{2-1}{2+1} = \frac{1}{3} = \frac{2}{2(1+2)}$$

Next, the induction step: We assume that

$$\prod_{k=2}^{n} \frac{k-1}{k+1} = \frac{2}{n(n+1)}$$

and we wish to prove

$$\prod_{k=2}^{n+1} \frac{k-1}{k+1} = \frac{2}{(n+1)(n+2)}$$

We start with the inductive assumption and multiply both sides by the $n+1$ term: $\dfrac{(n+1)-1}{(n+1)+1}$:

$$\left(\prod_{k=2}^{n} \frac{k-1}{k+1}\right) \cdot \frac{(n+1)-1}{(n+1)+1} = \frac{2}{n(n+1)} \cdot \frac{(n+1)-1}{(n+1)+1}$$
$$\prod_{k=2}^{n+1} \frac{k-1}{k+1} = \frac{2}{n(n+1)} \cdot \frac{n}{n+2}$$
$$\prod_{k=2}^{n+1} \frac{k-1}{k+1} = \frac{2n}{n(n+1)(n+2)}$$
$$\prod_{k=2}^{n+1} \frac{k-1}{k+1} = \frac{2}{(n+1)(n+2)}$$

We have thus proven the inductive step, and the proof is finished.

4. Prove $\forall\, n, m, l \in \mathbb{Z}$, if $n \mid m+l$ and $n \mid m$, then $n \mid l$.

We prove this directly and assuming the hypothesis gives $\exists\, k_1, k_2 \in \mathbb{Z}$ such that $m + l = n \cdot k_1$ and $m = n \cdot k_2$. Therefore $n \cdot k_2 + l = n \cdot k_1$. Solving for $l$ is this equation gives $l = n \cdot (k_1 - k_2)$. Setting $k = k_1 - k_2 \in \mathbb{Z}$ we have $l = n \cdot k$, thus $n \mid l$.

5. Prove $\forall\, m \in \mathbb{Z}$, $m$ is odd iff $4m^2 - 5m + 2$ is odd.

We prove two statements to prove the biconditional:

(A) If $m$ is odd, then $4m^2 - 5m + 2$ is odd.
(B) If $4m^2 - 5m + 2$ is odd, then $m$ is odd.

For (A) we prove directly: Assuming $m$ is odd, then $\exists\, k \in \mathbb{Z}$ such that $m = 2k + 1$. Thus

$$\begin{aligned} 4m^2 - 5m + 2 &= 4(2k+1)^2 - 5(2k+1) + 2 \\ &= 4(4k^2 + 4k + 1) - 10k - 5 + 2 \\ &= 16k^2 + 16k + 4 - 10k - 3 \\ &= 16k^2 + 6k + 1 \\ &= 2(8k^2 + 3k) + 1 \\ &= 2l + 1 \end{aligned}$$

where $l = 8k^2 + 3k \in \mathbb{Z}$, thus $4m^2 - 5m + 2$ is odd.

For (B), we prove the contrapositive statement: If $m$ is even, then $4m^2 - 5m + 2$ is even.

Assuming $m$ is even, then $\exists\, n \in \mathbb{Z}$ such that $m = 2n$. Thus

$$\begin{aligned} 4m^2 - 5m + 2 &= 4(2n)^2 - 5(2n) + 2 \\ &= 4(4n^2) - 10n + 2 \\ &= 16n^2 - 10n + 2 \\ &= 2(8n^2 - 5n + 1) \\ &= 2p \end{aligned}$$

where $p = 8n^2 - 5n + 1 \in \mathbb{Z}$, thus $4m^2 - 5m + 2$ is even. We have proven both conditional sentences, thus the original biconditional statement has been proven.