Math 3283 - Foundations of Mathematics

Final - 2021.12.08

Name: ______________________________

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1. Prove by induction that for $x > 0$, the inequality $(1+x)^n \geq 1 + nx$ holds for $n \geq 0$.

2. Construct a truth table for the sentence $(p \to q) \to [(p \to r) \to (p \to (q \wedge r))]$

3. State the inverse to the sentence: "If today is Wednesday and it is the finals week, then we are taking our Foundations final exam. "

4. For arbitrary sets A, B, and C, prove that $A \cup B \subseteq C \longleftrightarrow A \subseteq C \wedge B \subseteq C$.

5. For arbitrary sets A, B, C, and D, prove that $(A \times C) - (B \times D) = (A \times (C - D)) \cup ((A - B) \times C)$.
Hint: assume $(x, y) \in (A \times C) - (B \times D)$ as your first step.

6. Remember that the definition of the greatest common divisor of two natural numbers m and n, denoted $\text{GCD}(m, n)$ is the largest integer r such that $r|m$ and $r|n$. Thus, as an example, $\text{GCD}(30, 105) = 15$. We will define the relation G on the set of natural numbers to be $mGn \longleftrightarrow \text{GCD}(m, n) > 1$. Thus, $30\,G\,105$ since $\text{GCD}(30, 105) = 15 > 1$, while $\sim 33\,G\,70$ since $\text{GCD}(33, 70) = 1$. Determine if the relation G on the set of natural numbers has any of the properties: reflexive, symmetric, antisymmetric, or transitive. If G does not have one of the properties given, provide a counterexample. *Note: You must check all four properties.*

7. For the set $A = \{0, 1, 2, 3\}$, perform the following:
   (a) Construct a partition of the set A.
   (b) Define the relation R on A based on the partition given in part (a).
   (c) Lastly, compute A/R.

8. Prove that for any $n \in \mathbb{Z}$, the expression $2n^2 + n + 1$ is not divisible by 3. *Hint: If you attempt a proof by cases, it will require three cases.*

9. What is the difference between a function's codomain and its range?

10. State the definition of a function $f(x)$ being surjective (onto).

11. Determine which of the following quantified statements are true. You may assume the universe of discourse is $\mathbb{R}$.
   (a) $\exists x, y,\ (x \cdot y > -1)$
   (b) $\exists x,\ \forall y,\ (x \cdot y > -1)$
   (c) $\forall x\, \exists y,\ (x \cdot y > -1)$
   (d) $\forall x, y,\ (x \cdot y > -1)$

12. Compute $\mathcal{P}(\{1, \{1\}, \{1, 2\}\})$

13. Prove that for any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

14. Prove that if $m, n \in \mathbb{Z}$ and $4|(m^2 + n^2)$, then m and n are not both odd.

15. Prove that there exists no positive integers p and q such that $10p + 32q = 145$.