Math 2283 - Honors Logic

Homework - Chapter 8

Name: _____

Axiom 1. $\forall x, y\ (x = y \vee x < y \vee x > y)$

Axiom 2. $x < y \to y \not< x$

Axiom 3. $x > y \to y \not> x$

Axiom 4. $(x < y \wedge y < z) \to x < z$

Axiom 5. $(x > y \wedge y > z) \to x > z$

Theorem 1. $x \not< x$

Theorem 2. $x \not> x$

Theorem 3. $x > y \longleftrightarrow y < x$

Theorem 4. $x \neq y \longleftrightarrow (x < y \vee y < x)$

Theorem 5. $x \neq y \longleftrightarrow (x > y \vee y > x)$

Theorem 6. *For any numbers $x, y \in \mathbb{R}$, exactly one of the three formulas: $x = y$, $x < y$ and $x > y$ is satisfied.*

Theorem 7. $x \leq y \leftrightarrow x \not> y$

Theorem 8. $x < y \leftrightarrow (x \leq y \wedge x \neq y)$

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1. Let the formula:

$$x \circledS y$$

express the fact that the numbers x and y satisfy one of the following conditions:

(i) the number x has a smaller absolute value than the number y,

(ii) if the absolute values of x and y are the same, x is negative and y is positive.

In logical symbolism, this is given by

$$x \circledS y \overset{def}{\longleftrightarrow} [\,(|x| < |y|) \vee (|x| = |y| \wedge (x < 0 \wedge y > 0)\,]$$

Further, let the relation $\circledR$ to be defined in terms of $\circledS$ as follows:

$$x \circledR y \overset{def}{\longleftrightarrow} y \circledS x.$$

Show, on the basis of arithmetic, that the set of all numbers and the relations $\circledR$ and $\circledS$ just defined constitute a model of the first group of axioms.

2. Establish the validity of the following general law of the theory of relations (cf. remarks made in Section 6.2):

Theorem 1′. *Every relation R which is asymmetrical in the class K is also irreflexive in that class.*

3. Prove the following:

Theorem 2′. *Every relation R which is irreflexive and transitive in the class K is also asymmetrical in that class.*

5. Derive the following theorems from Axiom 4 and Definition 1:

Theorem D. $(x < y \wedge y \leq z) \to x < z$

Theorem E. $(x \leq y \wedge y < z) \to x < z$

Theorem F. $(x \leq y \wedge y < z \wedge z \leq t) \to x < t$

6. Show that the relations $\leq$ and $\geq$ are reflexive, transitive, and connected. Are these relations symmetrical or asymmetrical?

7. Between which of the relations: $=$, $<$, $>$, $\neq$, $\leq$, and $\geq$, does the relation of inclusion hold?